

Multiplicity results of fractional p -Laplace equations with sign-changing and singular nonlinearity

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Abstract

In this article, we study the following fractional p -Laplacian equation with singular nonlinearity

$$(P_\lambda) \begin{cases} -2 \int_{\mathbb{R}^n} \frac{|w(y)-w(x)|^{p-2}(w(y)-w(x))}{|x-y|^{n+ps}} dy = a(x)w^{-q} + \lambda b(x)w^r & \text{in } \Omega \\ w > 0 \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$, $n > ps, s \in (0, 1)$, $\lambda > 0$, $0 < q < 1$, $q < p - 1 < r < p_s^* - 1$ with $p_s^* = \frac{np}{n-ps}$, $a : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that $0 < a(x) \in L^{\frac{p_s^*}{p_s^*-1+q}}(\Omega)$, and $b : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a sign-changing function such that $b(x) \in L^{\frac{p_s^*}{p_s^*-1-r}}(\Omega)$. Using variational methods, we show existence and multiplicity of positive solutions of (P_λ) with respect to the parameter λ .

Key words: Non-local operator, singular nonlinearity, sign-changing weight function, Variational methods.

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1 Introduction

Let $s \in (0, 1)$ and let $0 \in \Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n > ps$. Then we consider the following problem with singular nonlinearity:

$$(P_\lambda) \begin{cases} -2 \int_{\mathbb{R}^n} \frac{|w(y)-w(x)|^{p-2}(w(y)-w(x))}{|y|^{n+ps}} dy = a(x)w^{-q} + \lambda b(x)w^r & \text{in } \Omega \\ w > 0 & \text{in } \Omega, \quad w = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

We assume the following assumptions on a and b :

(a1) $a : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that $0 < a \in L^{\frac{p_s^*}{p_s^*-1+q}}(\Omega)$.

(b1) $b : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a sign-changing function such that $b^+ \not\equiv 0$ and $b(x) \in L^{\frac{p_s^*}{p_s^*-1-r}}(\Omega)$.

Also $\lambda > 0$ is a parameter and $0 < q < 1$, $q < p - 1 < r < p_s^* - 1$, with $p_s^* = \frac{np}{n-ps}$, known as fractional critical Sobolev exponent.

The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusions in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, one can see [3, 18] and reference therein. Recently the fractional elliptic equation attracts a lot of interest in nonlinear analysis such as in [7, 31, 32, 33, 34]. Caffarelli and Silvestre [7] gave a new formulation of fractional Laplacian through Dirichlet-Neumann maps. This is commonly used in the literature since it allows us to write a nonlocal problem to a local problem which allow us to use the variational methods to study the existence and uniqueness.

On the other hand, the fractional elliptic problem have been investigated by many authors, for example, [31, 32] for subcritical case, [33, 34] for critical case with polynomial type nonlinearities. Moreover, by Nehari manifold and fibering maps, the author obtained the existence of multiple solutions for fractional equations for critical [36] and subcritical case [20, 21] and reference therein. In case of square root of Laplacian, existence and multiplicity results for sublinear and superlinear type of nonlinearity with sign-changing weight functions is studied in [35]. In [35], author used the idea of Caffarelli and Silvestre [7], which gives a formulation of the fractional Laplacian through Dirichlet-Neumann maps. Also in case of fractional p -Laplacian, existence and multiplicity results for polynomial type nonlinearities is studied by many authors see [20, 21, 24, 25, 29] and reference therein. Also eigenvalue problem related to p -fractional Laplacian is studied in [16, 28].

For $s = 1$, the paper by Crandall, Robinowitz and Tartar [10] is the starting point on semi-linear problem with singular nonlinearity. There is a large literature on singular nonlinearity see [1, 2, 10, 11, 12, 13, 14, 15, 19, 22, 23, 26, 27] and reference therein. In [9], Chen showed the existence and multiplicity of the following problem

$$\begin{cases} -\Delta w - \frac{\lambda}{|x|^2} w = \frac{f(x)}{w^q} + \mu g(x)w^p & \text{in } \Omega \setminus \{0\} \\ w > 0 & \text{in } \Omega \setminus \{0\}, \quad w = 0 & \text{in } \partial\Omega. \end{cases}$$

where $0 \in \Omega$ is a bounded smooth domain of \mathbb{R}^n with smooth boundary, $0 < \lambda < \frac{(n-2)^2}{4}$, $0 < q < 1 < p < \frac{n+2}{n-2}$, $f(x) > 0$ and g is sign-changing continuous function.

In [17], Fang proved the existence of solution of the following singular problem

$$(-\Delta)^s w = w^{-p}, \quad w > 0 \text{ in } \Omega, w = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

with $0 < p < 1$, using the method of sub and super solution. Recently, in [4], Barrios, Peral and et al. extend the result of [17]. They studied the existence result for the following fractional equation with singular type nonlinearities

$$\begin{cases} (-\Delta)^s w = \lambda \frac{f(x)}{w^\gamma} + Mw^p & \text{in } \Omega \\ w > 0 & \text{in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

where Ω is a bounded smooth domain of \mathbb{R}^n , $n > 2s$, $0 < s < 1$, $M \in \{0, 1\}$, $\gamma > 0$, $\lambda > 0$, $p > 1$ and $f \in L^m(\Omega)$, $m \geq 1$ is a nonnegative function. For $M = 0$, they proved the existence of solution for every $\gamma > 0$ and $\lambda > 0$. For $M = 1$ and $f \equiv 1$, they showed that there exist Λ such that it has a solution for every $0 < \lambda < \Lambda$, and have no solution for $\lambda > \Lambda$.

To the best of our knowledge, there is no work related to fractional p -Laplacian with singular and sign-changing nonlinearity. In this work, we studied the multiplicity results for fractional p -Laplacian equation with singular nonlinearity and sign-changing weight function with respect to the parameter λ . This work is motivated by the work of Chen and Chen in [9]. But one can not directly extend all the results for fractional p -Laplacian, due to the non-local behavior of the operator and the bounded support of the test function is not preserved. Also due to the singularity of the problem, the associated functional is not differentiable in the sense of Gâteaux. The results obtained here are somehow expected but we show how the results arise out of nature of the Nehari manifold.

The paper is organized as follows: Section 2 is devoted to some preliminaries and notations. we also state our main results. In section 3, we study the decomposition of Nehari manifold and the associated energy functional is bounded below and coercive. Section 3 contains the existence of a nontrivial solutions in \mathcal{N}_λ^+ and \mathcal{N}_λ^- .

We will use the following notation throughout this paper: $\|a\|$, $\|b\|$ denote the norm in $L^{\frac{p_s^*}{p_s^*-1+q}}(\Omega)$, $L^{\frac{p_s^*}{p_s^*-1-r}}(\Omega)$ respectively.

2 Preliminaries:

In this section we give some definitions and functional settings. At the end of this section, we state our main results. For this we define $W^{s,p}(\Omega)$, the usual fractional Sobolev space $W^{s,p}(\Omega) := \left\{ w \in L^p(\Omega); \frac{(w(x)-w(y))^p}{|x-y|^{\frac{p}{p}+s}} \in L^p(\Omega \times \Omega) \right\}$ endowed with the norm

$$\|w\|_{W^{s,p}(\Omega)} = \|w\|_{L^p} + \left(\int_{\Omega \times \Omega} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}}. \quad (2.1)$$

To study fractional Sobolev space in details we refer [30].

Due to the non-localness of the operator, we define linear space as follows:

$$X = \left\{ w \mid w : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is measurable, } w|_{\Omega} \in L^p(\Omega) \text{ and } \frac{w(x) - w(y)}{|x - y|^{\frac{n+ps}{p}}} \in L^p(Q) \right\}$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. In case of $p = 2$, the space X was firstly introduced by Servadei and Valdinoci [31]. The space X is a normed linear space endowed with the norm

$$\|w\|_X = \|w\|_{L^p(\Omega)} + \left(\int_Q \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}}. \quad (2.2)$$

Then we define

$$X_0 = \{w \in X : w = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

with the norm

$$\|w\| = \left(\int_Q \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}} \quad (2.3)$$

is a reflexive Banach space. We notice that, the norms in (2.1) and (2.2) are not same because $\Omega \times \Omega$ is strictly contained in Q . Now we define the space

$$C_{X_0} := \{w \in C_c^\infty(\mathbb{R}^n) : w = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

Then C_{X_0} is a dense in the space X_0 .

$$\text{Define } S := \inf_{w \in X_0} \left\{ \frac{\int_{\mathbb{R}^{2n}} |w(x) - w(y)|^p |x - y|^{-(n+ps)} dx dy}{\left(\int_{\mathbb{R}^n} |u|^{ps} dx \right)^{\frac{p}{s}}} \right\}.$$

Definition 2.1 A weak solution of the problem (P_λ) is a function $w \in X_0$, $w > 0$ in Ω such that for every $v \in X_0$

$$\int_Q \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (v(x) - v(y))}{|x - y|^{(n+ps)}} dx dy = \int_{\Omega} a(x) (w^{-q} v)(x) dx + \lambda \int_{\Omega} b(x) (w^r v)(x) dx.$$

In order to present the existence of positive solution of (P_λ) , we will consider the following problem

$$(P_\lambda^+) \begin{cases} -2 \int_{\mathbb{R}^n} \frac{|w(y) - w(x)|^{p-2} (w(y) - w(x))}{|x - y|^{n+ps}} dy = a(x) w_+^{-q} + \lambda b(x) w_+^r & \text{in } \Omega \\ w > 0 & \text{in } \Omega, \quad w = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

where $w_+ := \max\{w, 0\}$, denote the positive part of w . Then the function $w \in X_0$, $w > 0$ in Ω is a weak solution of the problem (P_λ^+) if for every $v \in X_0$

$$\int_Q \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (v(x) - v(y))}{|x - y|^{(n+ps)}} dx dy = \int_{\Omega} a(x) (w_+^{-q} v)(x) dx + \lambda \int_{\Omega} b(x) (w_+^r v)(x) dx.$$

We note that if $w > 0$ is a solution of (P_λ^+) then one can easily see that w is also a solution (P_λ) . To find the solution of (P_λ^+) , we will use variational approach. So we define the associated functional $J_\lambda : X_0 \rightarrow [-\infty, \infty)$ as

$$J_\lambda(w) = \frac{1}{p} \int_Q \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy - \frac{1}{1-q} \int_\Omega a(x) w_+^{1-q}(x) dx - \frac{\lambda}{r+1} \int_\Omega b(x) w_+^{r+1}(x) dx.$$

Now for $w \in X_0$, we define the fiber map $\phi_w : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\phi_w(t) = J_\lambda(tw) = \frac{t^p}{p} \|w\|^p - \frac{t^{1-q}}{1-q} \int_\Omega a(x) w_+^{1-q}(x) dx - \frac{\lambda t^{r+1}}{r+1} \int_\Omega b(x) w_+^{r+1}(x) dx.$$

Also

$$\begin{aligned} \phi'_w(t) &= t^{p-1} \|w\|^p - t^{-q} \int_\Omega a(x) w_+^{1-q}(x) dx - \lambda t^r \int_\Omega b(x) w_+^{r+1}(x) dx, \\ \phi''_w(t) &= (p-1)t^{p-2} \|w\|^p + qt^{-q-1} \int_\Omega a(x) w_+^{1-q}(x) dx - r\lambda t^{r-1} \int_\Omega b(x) w_+^{r+1}(x) dx. \end{aligned}$$

It is easy to see that the energy functional J_λ is not bounded below on the space X_0 . But we will show that it is bounded below on an appropriate subset of X_0 and a minimizer on subsets of this set gives rise to solutions of (P_λ^+) . In order to obtain the existence results, we define

$$\begin{aligned} \mathcal{N}_\lambda &:= \{w \in X_0 : \phi'_w(t) = \langle J'_\lambda(w), w \rangle = 0\} \\ &= \left\{ w \in X_0 : \|w\|^p = \int_\Omega a(x) w_+^{1-q}(x) dx + \lambda \int_\Omega b(x) w_+^{r+1}(x) dx \right\}. \end{aligned}$$

Note that $w \in \mathcal{N}_\lambda$ if w is a solution of problem (P_λ^+) . Also one can easily see that $tw \in \mathcal{N}_\lambda$ if and only if $\phi'_w(t) = 0$. In order to obtain our result, we decompose \mathcal{N}_λ with $\mathcal{N}_\lambda^\pm, \mathcal{N}_\lambda^0$ defined as follows:

$$\begin{aligned} \mathcal{N}_\lambda^\pm &:= \{w \in \mathcal{N}_\lambda : \phi''_w(1) \geq 0\} = \left\{ w \in \mathcal{N}_\lambda : (p-1+q)\|w\|^p \geq \lambda(r+q) \int_\Omega b(x) w_+^{r+1}(x) dx \right\} \\ \mathcal{N}_\lambda^0 &:= \{w \in \mathcal{N}_\lambda : \phi''_w(1) = 0\} = \left\{ w \in \mathcal{N}_\lambda : (p-1+q)\|w\|^p = \lambda(r+q) \int_\Omega b(x) w_+^{r+1}(x) dx \right\}. \end{aligned}$$

Our results are as follows:

Inspired by [9], we show that how variational methods can be used to established some existence and multiplicity results for (P_λ^+) :

Theorem 2.2 *Suppose that $\lambda \in (0, \Lambda)$, where*

$$\Lambda := \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{1}{\|b\|} \left(\frac{S^{r+q}}{\|a\|^{r-p+1}} \right)^{\frac{1}{p-1+q}}$$

then the problem (P_λ) has at least two solutions $w \in \mathcal{N}_\lambda^+$, $W \in \mathcal{N}_\lambda^-$ with $\|W\| > \|w\|$.

Next, we obtain the blow up behavior of the solution $W_\epsilon \in \mathcal{N}_\lambda^-$ of problem (P_λ) with $r = p - 1 + \epsilon$ as $\epsilon \rightarrow 0^+$.

Theorem 2.3 *let $W_\epsilon \in \mathcal{N}_\lambda^-$ be the solution of problem (P_λ) with $r = p - 1 + \epsilon$, where $\lambda \in (0, \Lambda)$, then*

$$\|W_\epsilon\| > C_\epsilon \left(\frac{\Lambda}{\lambda} \right)^{\frac{1}{\epsilon}},$$

where

$$C_\epsilon = \left(1 + \frac{p-1+q}{\epsilon} \right)^{\frac{1}{p-1+q}} \|a\|^{\frac{1}{p-1+q}} \left(\frac{1}{\sqrt[p]{S}} \right)^{\frac{1-q}{p-1+q}} \rightarrow \infty \text{ as } \epsilon \rightarrow 0^+.$$

Namely, W_ϵ blows up faster than exponentially with respect to ϵ .

Remark: If w is a positive solution of the following problem

$$\begin{cases} -2 \int_{\mathbb{R}^n} \frac{|w(y)-w(x)|^{p-2}(w(y)-w(x))}{|y|^{n+ps}} dy = a(x)w^{-q} + \lambda b(x)w^r & \text{in } \Omega \\ w > 0 & \text{in } \Omega, \quad w = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

then one can easily see that $u = \lambda^{\frac{1}{r-1+p}} w$ is a positive solution of the following problem

$$(Q_\lambda) \begin{cases} -2 \int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y|^{n+ps}} dy = \lambda^{\frac{p-1+q}{r-p+1}} a(x)u^{-q} + b(x)u^r & \text{in } \Omega \\ u > 0 & \text{in } \Omega, \quad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

That is, the problem (Q_λ) has two positive solutions for $\lambda \in (0, \Lambda)$.

3 Fiber map analysis

In this section, we show that \mathcal{N}_λ^\pm is nonempty and $\mathcal{N}_\lambda^0 = \{0\}$. Moreover, J_λ is bounded below and coercive.

Lemma 3.1 *Let $\lambda \in (0, \Lambda)$. Then for each $w \in X_0$ with $\int_\Omega a(x)w_+^{1-q}(x)dx > 0$, we have the following:*

- (i) $\int_\Omega b(x)w_+^{r+1}(x)dx \leq 0$, then there exists a unique $0 < t_1 < t_{max}$ such that $t_1 w \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1 w) = \inf_{t>0} J_\lambda(tw)$,
- (ii) $\int_\Omega b(x)w_+^{r+1}(x)dx > 0$, then there exists a unique t_1 and t_2 with $0 < t_1 < t_{max} < t_2$ such that $t_1 w \in \mathcal{N}_\lambda^+$, $t_2 w \in \mathcal{N}_\lambda^-$ and $J_\lambda(t_1 w) = \inf_{0 \leq t \leq t_{max}} J_\lambda(tw)$, $J_\lambda(t_2 w) = \sup_{t \geq t_1} J_\lambda(tw)$.

Proof. For $t > 0$, we define

$$\psi_w(t) = t^{p-1-r} \|w\|^p - t^{-r-q} \int_\Omega a(x)w_+^{1-q}(x)dx - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx.$$

One can easily see that $\psi_w(t) \rightarrow -\infty$ as $t \rightarrow 0^+$. Now

$$\psi'_w(t) = (p-1-r)t^{p-2-r}\|w\|^p + (r+q)t^{-r-q-1} \int_{\Omega} a(x)w_+^{1-q}(x)dx.$$

$$\psi''_w(t) = (p-1-r)(p-2-r)t^{p-r-3}\|w\|^p - (r+q)(r+q+1)t^{-r-q-2} \int_{\Omega} a(x)w_+^{1-q}(x)dx.$$

Then $\psi'_w(t) = 0$ if and only if $t = t_{max} := \left[\frac{(r-p+1)\|w\|^p}{(r+q) \int_{\Omega} a(x)w_+^{1-q}(x)dx} \right]^{-\frac{1}{p-1+q}}$. Also

$$\begin{aligned} \psi''_w(t_{max}) &= (p-1-r)(p-2-r) \left[\frac{(r-p+1)\|w\|^p}{(r+q) \int_{\Omega} a(x)w_+^{1-q}(x)dx} \right]^{\frac{r-p+3}{p-1+q}} \|w\|^p \\ &\quad - (r+q)(r+q+1) \left[\frac{(r-p+1)\|w\|^p}{(r+q) \int_{\Omega} a(x)w_+^{1-q}(x)dx} \right]^{\frac{r+q+2}{p-1+q}} \int_{\Omega} a(x)w_+^{1-q}(x)dx \\ &= -\|w\|^p(r-p+1)(p-1+q) \left[\frac{(r-p+1)\|w\|^p}{(r+q) \int_{\Omega} a(x)w_+^{1-q}(x)dx} \right]^{\frac{r-p+3}{p-1+q}} < 0. \end{aligned}$$

Thus ψ_w achieves its maximum at $t = t_{max}$. Now using the Hölder's inequality and Sobolev inequality, we obtain

$$\begin{aligned} \int_{\Omega} a(x)w_+^{1-q}(x)dx &\leq \left[\int_{\Omega} |a(x)|^{\frac{p_s^*}{p_s^*-1+q}} \right]^{\frac{p_s^*+q-1}{p_s^*}} \left[\int_{\Omega} |w(x)|^{p_s^*} dx \right]^{\frac{1-q}{p_s^*}} \\ &\leq \|a\| \left(\frac{\|w\|}{\sqrt[p]{S}} \right)^{1-q}. \end{aligned} \quad (3.1)$$

$$\begin{aligned} \int_{\Omega} b(x)w_+^{r+1}(x)dx &\leq \left[\int_{\Omega} |b(x)|^{\frac{p_s^*}{p_s^*-1-r}} \right]^{\frac{p_s^*-r-1}{p_s^*}} \left[\int_{\Omega} |w(x)|^{p_s^*} dx \right]^{\frac{r+1}{p_s^*}} \\ &\leq \|b\| \left(\frac{\|w\|}{\sqrt[p]{S}} \right)^{r+1}. \end{aligned} \quad (3.2)$$

Using (3.1) and (3.2) we obtain,

$$\begin{aligned} \psi_w(t_{max}) &= \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{\|w\|^{\frac{p(r+q)}{p-1+q}}}{\left[\int_{\Omega} a(x)w_+^{1-q}(x)dx \right]^{\frac{r-p+1}{p-1+q}}} - \lambda \int_{\Omega} b(x)w_+^{r+1}(x)dx \\ &\geq \left[\frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \left(\frac{(\sqrt[p]{S})^{(1-q)}}{\|a\|} \right)^{\frac{(r-p+1)}{p-1+q}} - \lambda \|b\| \left(\frac{1}{\sqrt[p]{S}} \right)^{r+1} \right] \|w\|^{r+1} \\ &\equiv E_{\lambda} \|w\|^{r+1}. \end{aligned} \quad (3.3)$$

where

$$E_{\lambda} = \left[\frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \left(\frac{(\sqrt[p]{S})^{(1-q)}}{\|a\|} \right)^{\frac{r-p+1}{p-1+q}} - \lambda \|b\| \left(\frac{1}{\sqrt[p]{S}} \right)^{r+1} \right]$$

Then we see that $E_\lambda = 0$ if and only if $\lambda = \Lambda$, where

$$\Lambda := \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{1}{\|b\|} \left(\frac{S^{r+q}}{\|a\|^{r-p+1}} \right)^{\frac{1}{p-1+q}}.$$

Thus for $\lambda \in (0, \Lambda)$, we have $E_\lambda > 0$, and therefore it follows from (3.3) that $\psi_w(t_{max}) > 0$.

(i) If $\int_\Omega b(x)w_+^{r+1}(x)dx \geq 0$, then $\psi_w(t) \rightarrow -\lambda \int_\Omega b(x)w_+^{r+1}(x)dx < 0$ as $t \rightarrow \infty$. Consequently, $\psi_w(t)$ has exactly two points $0 < t_1 < t_{max} < t_2$ such that

$$\psi_w(t_1) = 0 = \psi_w(t_2) \text{ and } \psi'_w(t_1) > 0 > \psi'_w(t_2).$$

Now we show that if $\psi_w(t) = 0$ and $\psi'_w(t) > 0$, then $tw \in \mathcal{N}_\lambda^+$.

$$\begin{aligned} \psi_w(t) = 0 &\Rightarrow t^{p-1-r}\|w\|^p - t^{-r-q} \int_\Omega a(x)w_+^{1-q}(x)dx - \lambda \int_\Omega b(x)w_+^{r+1}(x)dx = 0 \\ &\Rightarrow \|tw\|^p = \int_\Omega a(x)(tw)_+^{1-q}(x)dx + \lambda \int_\Omega b(x)(tw)_+^{r+1}(x)dx \\ &\Rightarrow tw \in \mathcal{N}_\lambda, \end{aligned}$$

and therefore

$$\begin{aligned} \psi'_w(t) > 0 &\Rightarrow (p-1-r)t^{p-2-r}\|w\|^p - (-r-q)t^{-r-q-1} \int_\Omega a(x)w_+^{1-q}(x)dx > 0 \\ &\Rightarrow (p-1-r)\|tw\|^p + (r+q) \int_\Omega a(x)(tw)_+^{1-q}(x)dx > 0 \\ &\Rightarrow (p-1-r)\|tw\|^p + (r+q) \left[\|tw\|^p - \lambda \int_\Omega b(x)(tw)_+^{r+1}(x)dx \right] > 0, \text{ since } tw \in \mathcal{N}_\lambda \\ &\Rightarrow (p-1+q)\|tw\|^p - \lambda(r+q) \int_\Omega b(x)(tw)_+^{r+1}(x)dx > 0 \\ &\Rightarrow tw \in \mathcal{N}_\lambda^+. \end{aligned}$$

Similarly one can show that if $\psi_w(t) = 0$ and $\psi'_w(t) < 0$, then $tw \in \mathcal{N}_\lambda^-$.

Now $\phi'_w(t) = t^r \psi_w(t)$. Thus $\phi'_w(t) < 0$ in $(0, t_1)$, $\phi'_w(t) > 0$ in (t_1, t_2) and $\phi'_w(t) < 0$ in (t_2, ∞) . Hence $J_\lambda(t_1w) = \inf_{0 \leq t \leq t_{max}} J_\lambda(tw)$, $J_\lambda(t_2w) = \sup_{t \geq t_1} J_\lambda(tw)$. Moreover $t_1w \in \mathcal{N}_\lambda^+$ and $t_2w \in \mathcal{N}_\lambda^-$.

(ii) If $\int_\Omega b(x)w_+^{r+1}(x)dx < 0$ and $\psi_w(t) \rightarrow -\lambda \int_\Omega b(x)w_+^{r+1}(x)dx > 0$ as $t \rightarrow \infty$. Consequently, $\psi_w(t)$ has exactly one point $0 < t_1 < t_{max}$ such that

$$\psi_w(t_1) = 0 \text{ and } \psi'_w(t_1) > 0.$$

Using $\phi'_w(t) = t^r \psi_w(t)$, we have $\phi'_w(t) < 0$ in $(0, t_1)$, $\phi'_w(t) > 0$ in (t_1, ∞) . So, $J_\lambda(t_1w) = \inf_{t \geq 0} J_\lambda(tw)$. Hence, it follows that $t_1w \in \mathcal{N}_\lambda^+$.

Corollary 3.2 Suppose that $\lambda \in (0, \Lambda)$, then $\mathcal{N}_\lambda^\pm \neq \emptyset$.

Proof. By (a1) and (b1), we can choose $w \in X_0 \setminus \{0\}$ such that $\int_\Omega a(x)w_+^{1-q}(x)dx > 0$ and $\int_\Omega b(x)w_+^{r+1}(x)dx > 0$. By (ii) of Lemma 3.1 there exists unique t_1 and t_2 such that $t_1w \in \mathcal{N}_\lambda^+$, $t_2w \in \mathcal{N}_\lambda^-$. In conclusion, $\mathcal{N}_\lambda^\pm \neq \emptyset$. \square

Lemma 3.3 For $\lambda \in (0, \Lambda)$, we have $\mathcal{N}_\lambda^0 = \{0\}$.

Proof. We prove this by contradiction. Assume that there exists $0 \neq w \in \mathcal{N}_\lambda^0$. Then it follows from $w \in \mathcal{N}_\lambda^0$ that

$$(p-1+q)\|w\|^p = \lambda(r+q) \int_{\Omega} b(x)w_+^{r+1}(x)dx$$

and consequently

$$\begin{aligned} 0 &= \|w\|^p - \int_{\Omega} a(x)w_+^{1-q}(x)dx - \lambda \int_{\Omega} b(x)w_+^{r+1}(x)dx \\ &= \|w\|^p - \int_{\Omega} a(x)w_+^{1-q}(x)dx - \frac{p-1+q}{r+q}\|w\|^p \\ &= \frac{(r-p+1)}{(r+q)}\|w\|^p - \int_{\Omega} a(x)w_+^{1-q}(x)dx. \end{aligned}$$

Therefore, as $\lambda \in (0, \Lambda)$ and $w \neq 0$, we use similar arguments as those in (3.3) to get

$$\begin{aligned} 0 &< E_\lambda \|w\|^{r+1} \\ &\leq \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{\|w\|^{\frac{p(r+q)}{p-1+q}}}{\left[\int_{\Omega} a(x)w_+^{1-q}(x)dx \right]^{\frac{r-p+1}{p-1+q}}} - \lambda \int_{\Omega} b(x)w_+^{r+1}(x)dx \\ &= \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{\|w\|^{\frac{p(r+q)}{p-1+q}}}{\left(\frac{r-p+1}{r+q} \|w\|^p \right)^{\frac{r-p+1}{p-1+q}}} - \frac{(p-1+q)}{(r+q)} \|w\|^p = 0, \end{aligned}$$

a contradiction. Hence $w = 0$. That is, $\mathcal{N}_\lambda^0 = \{0\}$. □

We note that Λ is also related to a gap structure in \mathcal{N}_λ :

Lemma 3.4 Suppose that $\lambda \in (0, \Lambda)$, then there exist a gap structure in \mathcal{N}_λ :

$$\|W\| > A_\lambda > A_0 > \|w\| \text{ for all } w \in \mathcal{N}_\lambda^+, W \in \mathcal{N}_\lambda^-,$$

where

$$A_\lambda = \left[\frac{(p-1+q)}{\lambda(r+q)\|b\|} (\sqrt[p]{S})^{r+1} \right]^{\frac{1}{r-p+1}} \text{ and } A_0 = \left[\frac{(r+q)}{(r-p+1)} \|a\| \left(\frac{1}{\sqrt[p]{S}} \right)^{1-q} \right]^{\frac{1}{p+q-1}}.$$

Proof. If $w \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, then

$$\begin{aligned} 0 &< (p-1+q)\|w\|^p - \lambda(r+q) \int_{\Omega} b(x)w_+^{r+1}(x)dx \\ &= (p-1+q)\|w\|^p - (r+q) \left[\|w\|^p - \int_{\Omega} a(x)w_+^{1-q}(x)dx \right] \\ &= (p-1-r)\|w\|^p + (r+q) \int_{\Omega} a(x)w_+^{1-q}(x)dx. \end{aligned}$$

Hence it follows from (3.1)

$$(r - p + 1)\|w\|^p < (r + q) \int_{\Omega} a(x)w_+^{1-q}(x)dx \leq (r + q)\|a\| \left(\frac{\|w\|}{\sqrt[p]{S}} \right)^{1-q}$$

which yields

$$\|w\| < \left[\frac{(r + q)\|a\|}{(r - p + 1)} \left(\frac{1}{\sqrt[p]{S}} \right)^{1-q} \right]^{\frac{1}{p+q-1}} \equiv A_0.$$

If $W \in \mathcal{N}_{\lambda}^{-}$, then it follows from (3.2) that

$$(p - 1 + q)\|W\|^p < \lambda(r + q) \int_{\Omega} b(x)W_+^{r+1}(x)dx \leq \lambda(r + q)\|b\| \left(\frac{\|W\|}{\sqrt[p]{S}} \right)^{r+1}$$

which yields

$$\|W\| > \left[\frac{(p - 1 + q)}{\lambda(r + q)\|b\|} (\sqrt[p]{S})^{r+1} \right]^{\frac{1}{r-p+1}} \equiv A_{\lambda}.$$

Now we show that $A_{\lambda} = A_0$ if and only if $\lambda = \Lambda$.

$$\begin{aligned} \lambda = \Lambda &= \frac{p - 1 + q}{r + q} \left(\frac{r - p + 1}{r + q} \right)^{\frac{r-p+1}{p-1+q}} \frac{1}{\|b\|} \left(\frac{S^{r+q}}{\|a\|^{r-p+1}} \right)^{\frac{1}{p-1+q}} \\ \Leftrightarrow A_{\lambda} &= \lambda^{-\frac{1}{r-p+1}} \left(\frac{p - 1 + q}{r + q} \right)^{\frac{1}{r-p+1}} \left(\frac{1}{\|b\|} \right)^{\frac{1}{r-p+1}} (\sqrt[p]{S})^{\frac{r+1}{r-p+1}} \\ &= \left(\frac{r + q}{r - p + 1} \right)^{\frac{1}{p-1+q}} \|a\|^{\frac{1}{p+q-1}} (\sqrt[p]{S})^{-\frac{p(r+q)}{(p-1+q)(r-p+1)} + \frac{r+1}{r-p+1}} = \left[\frac{(r + q)\|a\|}{(r - p + 1)(\sqrt[p]{S})^{1-q}} \right]^{\frac{1}{p+q-1}} \equiv A_0. \end{aligned}$$

Thus for all $\lambda \in (0, \Lambda)$, we can conclude that

$$\|W\| > A_{\lambda} > A_0 > \|w\| \text{ for all } w \in \mathcal{N}_{\lambda}^{+}, W \in \mathcal{N}_{\lambda}^{-}.$$

This completes the proof of the Lemma. \square

Lemma 3.5 Suppose that $\lambda \in (0, \Lambda)$, then $\mathcal{N}_{\lambda}^{-}$ is a closed set in X_0 - topology.

Proof. Let $\{W_k\}$ be a sequence in $\mathcal{N}_{\lambda}^{-}$ with $W_k \rightarrow W$ in X_0 . Then we have

$$\begin{aligned} \|W\|^p &= \lim_{k \rightarrow \infty} \|W_k\|^p \\ &= \lim_{k \rightarrow \infty} \left[\int_{\Omega} a(x)(W_k)_+^{1-q}(x)dx + \lambda \int_{\Omega} b(x)(W_k)_+^{r+1}(x)dx \right] \\ &= \int_{\Omega} a(x)W_+^{1-q}(x)dx + \lambda \int_{\Omega} b(x)W_+^{r+1}(x)dx \end{aligned}$$

and

$$\begin{aligned} (p - 1 + q)\|W\|^p - \lambda(r + q) \int_{\Omega} b(x)W_+^{r+1}(x)dx \\ = \lim_{k \rightarrow \infty} \left[(p - 1 + q)\|W_k\|^p - \lambda(r + q) \int_{\Omega} b(x)(W_k)_+^{r+1}(x)dx \right] \leq 0, \end{aligned}$$

i.e. $W \in \mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^0$. Since $\{W_k\} \subset \mathcal{N}_\lambda^-$, from Lemma 3.4 we have

$$\|W\| = \lim_{k \rightarrow \infty} \|W_k\| \geq A_0 > 0,$$

that is, $W \neq 0$. It follows from Lemma 3.1, that $W \notin \mathcal{N}_\lambda^0$ for any $\lambda \in (0, \Lambda)$. Thus $W \in \mathcal{N}_\lambda^-$. That is, \mathcal{N}_λ^- is a closed set in X_0 -topology for any $\lambda \in (0, \Lambda)$. \square

Lemma 3.6 *Let $w \in \mathcal{N}_\lambda^\pm$, then for any $\phi \in C_{X_0}$, there exists a number $\epsilon > 0$ and a continuous function $f : B_\epsilon(0) := \{v \in X_0 : \|v\| < \epsilon\} \rightarrow \mathbb{R}^+$ such that*

$$f(v) > 0, f(0) = 1 \text{ and } f(v)(w + v\phi) \in \mathcal{N}_\lambda^\pm \text{ for all } v \in B_\epsilon(0).$$

Proof. We give the proof only for the case $w \in \mathcal{N}_\lambda^+$, the case \mathcal{N}_λ^- may be preceded exactly. For any C_{X_0} , we define $F : X_0 \times \mathbb{R}^+ \rightarrow \mathbb{R}$ as follows:

$$F(v, t) = t^{p-1+q} \|w + v\phi\|^p - \int_\Omega a(x)(w + v\phi)_+^{1-q}(x) dx - \lambda t^{r+q} \int_\Omega b(x)(w + v\phi)_+^{r+1}(x) dx$$

Since $w \in \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda)$, we have that

$$F(0, 1) = \|w\|^p - \int_\Omega a(x)w_+^{1-q}(x) dx - \lambda \int_\Omega b(x)w_+^{r+1}(x) dx = 0,$$

and

$$\frac{\partial F}{\partial t}(0, 1) = (p-1+q)\|w\|^p - \lambda(r+q) \int_\Omega b(x)w_+^{r+1}(x) dx > 0.$$

Applying the implicit function Theorem at the point $(0, 1)$, we have that there exists $\bar{\epsilon} > 0$ such that for $\|v\| < \bar{\epsilon}$, $v \in X_0$, the equation $F(v, t) = 0$ has a unique continuous solution $t = f(v) > 0$. It follows from $F(0, 1) = 0$ that $f(0) = 1$ and from $F(v, f(v)) = 0$ for $\|v\| < \bar{\epsilon}$, $v \in X_0$ that

$$\begin{aligned} 0 &= f^{p-1+q}(v) \|w + v\phi\|^p - \int_\Omega a(x)(w + v\phi)_+^{1-q}(x) dx - \lambda f^{r+q}(v) \int_\Omega b(x)(w + v\phi)_+^{r+1}(x) dx \\ &= \frac{\|f(v)(w + v\phi)\|^p - \int_\Omega a(x)(f(v)(w + v\phi))_+^{1-q}(x) dx - \lambda \int_\Omega b(x)(f(v)(w + v\phi))_+^{r+1}(x) dx}{f^{1-q}(v)} \end{aligned}$$

that is,

$$f(v)(w + v\phi) \in \mathcal{N}_\lambda \text{ for all } v \in X_0, \|v\| < \bar{\epsilon}.$$

Since $\frac{\partial F}{\partial t}(0, 1) > 0$ and

$$\begin{aligned} \frac{\partial F}{\partial t}(v, f(v)) &= (p-1+q)f^{p-2+q}(v) \|w + v\phi\|^p - \lambda(r+q)f^{r+q-1}(v) \int_\Omega b(x)(w + v\phi)_+^{r+1}(x) dx \\ &= \frac{(p-1+q)\|f(v)(w + v\phi)\|^p - \lambda(r+q) \int_\Omega b(x)(f(v)(w + v\phi))_+^{r+1}(x) dx}{f^{2-q}(v)} \end{aligned}$$

we can take $\epsilon > 0$ possibly smaller ($\epsilon < \bar{\epsilon}$) such that for any $v \in X_0$, $\|v\| < \epsilon$,

$$(p-1+q)\|f(v)(w + v\phi)\|^p - \lambda(r+q) \int_\Omega b(x)(f(v)(w + v\phi))_+^{r+1}(x) dx > 0,$$

that is,

$$f(v)(w + v\phi) \in \mathcal{N}_\lambda^+ \text{ for all } v \in B_\epsilon(0).$$

This completes the proof of Lemma. \square

Lemma 3.7 J_λ is bounded below and coercive on \mathcal{N}_λ .

Proof. For $w \in \mathcal{N}_\lambda$, we obtain from (3.1) that

$$\begin{aligned} J_\lambda(w) &= \left(\frac{1}{p} - \frac{1}{r+1} \right) \|w\|^p - \left(\frac{1}{1-q} - \frac{1}{r+1} \right) \int_\Omega a(x) w_+^{1-q}(x) dx \\ &\geq \left(\frac{1}{p} - \frac{1}{r+1} \right) \|w\|^p - \left(\frac{1}{1-q} - \frac{1}{r+1} \right) \|a\| \left(\frac{\|w\|}{\sqrt[p]{S}} \right)^{1-q}. \end{aligned} \quad (3.4)$$

Now consider the function $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}$ as $\rho(t) = \alpha t^p - \beta t^{1-q}$, where α, β are both positive constants. One can easily show that ρ is convex ($\rho''(t) > 0$ for all $t > 0$) with $\rho(t) \rightarrow 0$ as $t \rightarrow 0$ and $\rho(t) \rightarrow \infty$ as $t \rightarrow \infty$. ρ achieves its minimum at $t_{min} = \left[\frac{\beta(1-q)}{p\alpha} \right]^{\frac{1}{p-1+q}}$ and

$$\rho(t_{min}) = \alpha \left[\frac{\beta(1-q)}{p\alpha} \right]^{\frac{p}{p-1+q}} - \beta \left[\frac{\beta(1-q)}{p\alpha} \right]^{\frac{1-q}{p-1+q}} = -\frac{(p-1+q)}{p} \beta^{\frac{p}{p-1+q}} \left(\frac{1-q}{p\alpha} \right)^{\frac{1-q}{p-1+q}}.$$

Applying $\rho(t)$ with $\alpha = \left(\frac{1}{p} - \frac{1}{r+1} \right)$, $\beta = \left(\frac{1}{1-q} - \frac{1}{r+1} \right) \|a\| \left(\frac{1}{\sqrt[p]{S}} \right)^{1-q}$ and $t = \|w\|$, $w \in \mathcal{N}_\lambda$, we obtain from (3.4) that

$$\lim_{\|w\| \rightarrow \infty} J_\lambda(w) \geq \lim_{t \rightarrow \infty} \rho(t) = \infty,$$

since $0 < q < 1 \leq p-1$. That is J_λ is coercive on \mathcal{N}_λ . Moreover it follows from (3.4) that

$$J_\lambda(w) \geq \rho(t) \geq \rho(t_{min}) \text{ (a constant)}, \quad (3.5)$$

i.e

$$J_\lambda(w) \geq -\frac{(p-1+q)}{p} \beta^{\frac{p}{p-1+q}} \left(\frac{1-q}{p\alpha} \right)^{\frac{1-q}{p-1+q}} = -\frac{(p-1+q)(r+1-p)}{(1-q)(r+1)} \left(\frac{r+q}{p(r+1-p)} \right)^{\frac{p}{p-1+q}}.$$

Thus J_λ is bounded below on \mathcal{N}_λ . □

4 Existence of Solutions in \mathcal{N}_λ^\pm

Now from Lemma 3.5, $\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ and \mathcal{N}_λ^- are two closed sets in X_0 provided $\lambda \in (0, \Lambda)$. Consequently, the Ekeland variational principle can be applied to the problem of finding the infimum of J_λ on both $\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ and \mathcal{N}_λ^- . First, consider $\{w_k\} \subset \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$ with the following properties:

$$J_\lambda(w_k) < \inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) + \frac{1}{k} \quad (4.1)$$

$$J_\lambda(w) \geq J_\lambda(w_k) - \frac{1}{k} \|w - w_k\|, \text{ for all } w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0. \quad (4.2)$$

Lemma 4.1 Show that the sequence $\{w_k\}$ is bounded in \mathcal{N}_λ . Moreover, there exists $0 \neq w \in X_0$ such that $w_k \rightharpoonup w$ weakly in X_0 .

Proof. From equations (3.5) and (4.1), we have

$$at^p - bt^{1-q} = \rho(t) \leq J_\lambda(w) < \inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) + \frac{1}{k} \leq C_5,$$

for sufficiently large k and a suitable positive constant. Hence putting $t = w_k$ in the above equation, we obtain $\{w_k\}$ is bounded.

Let $\{w_k\}$ is bounded in X_0 . Then, there exists a subsequence of $\{w_k\}_k$, still denoted by $\{w_k\}_k$ and $w \in X_0$ such that $w_k \rightharpoonup w$ weakly in X_0 , $w_k(\cdot) \rightarrow w(\cdot)$ strongly in $L^r(\Omega)$ for $1 \leq r < p_s^*$ and $w_k(\cdot) \rightarrow w(\cdot)$ a.e. in Ω .

For any $w \in \mathcal{N}_\lambda^+$, we have from $0 < q < 1 \leq p - 1 < r$ that

$$\begin{aligned} J_\lambda(w) &= \left(\frac{1}{p} - \frac{1}{1-q} \right) \|w\|^p + \left(\frac{1}{1-q} - \frac{1}{r+1} \right) \lambda \int_\Omega b(x) w_+^{r+1}(x) dx \\ &< \left(\frac{1}{p} - \frac{1}{1-q} \right) \|w\|^p + \left(\frac{1}{1-q} - \frac{1}{r+1} \right) \frac{p-1+q}{r+q} \|w\|^p \\ &= \left(\frac{1}{r+1} - \frac{1}{p} \right) \frac{(p-1+q)}{(1-q)} \|w\|^p < 0, \end{aligned}$$

which means that $\inf_{\mathcal{N}_\lambda^+} J_\lambda < 0$. Now for $\lambda \in (0, \Lambda)$, we know from Lemma 3.1, that $\mathcal{N}_\lambda^0 = \{0\}$. Together, these imply that $w_k \in \mathcal{N}_\lambda^+$ for k large and

$$\inf_{w \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda(w) = \inf_{w \in \mathcal{N}_\lambda^+} J_\lambda(w) < 0.$$

Therefore, by weak lower semi-continuity of norm,

$$J_\lambda(w) \leq \liminf_{k \rightarrow \infty} J_\lambda(w_k) = \inf_{\mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0} J_\lambda < 0,$$

that is, $w \not\equiv 0$ and $w \in X_0$. □

Lemma 4.2 Suppose $w_k \in \mathcal{N}_\lambda^+$ such that $w_k \rightharpoonup w$ weakly in X_0 . Then for $\lambda \in (0, \Lambda)$,

$$(p-1+q) \int_\Omega a(x) w_+^{1-q}(x) dx - \lambda(r-p+1) \int_\Omega b(x) w_+^{r+1}(x) dx > 0. \quad (4.3)$$

Moreover, there exists a constant $C_2 > 0$ such that

$$(p-1+q) \|w_k\|^p - \lambda(r+q) \int_\Omega b(x) (w_k)_+^{r+1}(x) dx \geq C_2 > 0. \quad (4.4)$$

Proof. For $\{w_k\} \subset \mathcal{N}_\lambda^+ (\subset \mathcal{N}_\lambda)$, we have

$$\begin{aligned} &(p-1+q) \int_\Omega a(x) w_+^{1-q}(x) dx - \lambda(r-p+1) \int_\Omega b(x) w_+^{r+1}(x) dx \\ &= \lim_{k \rightarrow \infty} \left[(p-1+q) \int_\Omega a(x) (w_k)_+^{1-q}(x) dx - \lambda(r-p+1) \int_\Omega b(x) (w_k)_+^{r+1}(x) dx \right] \\ &= \lim_{k \rightarrow \infty} \left[(p-1+q) \|w_k\|^p - \lambda(r+q) \int_\Omega b(x) (w_k)_+^{r+1}(x) dx \right] \geq 0. \end{aligned}$$

Now, we can argue by a contradiction and assume that

$$(p-1+q) \int_{\Omega} a(x) w_+^{1-q}(x) dx - \lambda(r-p+1) \int_{\Omega} b(x) w_+^{r+1}(x) dx = 0. \quad (4.5)$$

Using $w_k \in \mathcal{N}_\lambda$, the weak lower semi continuity of norm and (4.5) we have that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \left[\|w_k\|^p - \int_{\Omega} a(x) (w_k)_+^{1-q}(x) dx - \lambda \int_{\Omega} b(x) (w_k)_+^{r+1}(x) dx \right] \\ &\geq \|w\|^p - \int_{\Omega} a(x) w_+^{1-q}(x) dx - \lambda \int_{\Omega} b(x) w_+^{r+1}(x) dx \\ &= \begin{cases} \|w\|^p - \lambda \frac{r+q}{p-1+q} \int_{\Omega} b(x) w_+^{r+1}(x) dx \\ \|w\|^p - \frac{r+q}{r-p+1} \int_{\Omega} a(x) w_+^{1-q}(x) dx. \end{cases} \end{aligned}$$

Thus for any $\lambda \in (0, \Lambda)$ and $w \neq 0$, by similar arguments as those in (3.3) we have that

$$\begin{aligned} 0 &< E_\lambda \|w\|^{r+1} \\ &\leq \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{\|w\|^{\frac{p(r+q)}{p-1+q}}}{\left[\int_{\Omega} a(x) w_+^{1-q}(x) dx \right]^{\frac{r-p+1}{p-1+q}}} - \lambda \int_{\Omega} b(x) w_+^{r+1}(x) dx \\ &= \frac{(p-1+q)}{(r+q)} \left(\frac{r-p+1}{r+q} \right)^{\frac{r-p+1}{p-1+q}} \frac{\|w\|^{\frac{p(r+q)}{p-1+q}}}{\left(\frac{r-p+1}{r+q} \|w\|^p \right)^{\frac{r-p+1}{p-1+q}}} - \frac{(p-1+q)}{(r+q)} \|w\|^p = 0, \end{aligned}$$

which is clearly impossible. Now by (4.3), we have that

$$(p-1+q) \int_{\Omega} a(x) (w_k)_+^{1-q}(x) dx - \lambda(r-p+1) \int_{\Omega} b(x) (w_k)_+^{r+1}(x) dx \geq C_2 \quad (4.6)$$

for sufficiently large k and a suitable positive constant C_2 . This, together with the fact that $w_k \in \mathcal{N}_\lambda$ we obtain equation (4.4). \square

Fix $\phi \in C_{X_0}$ with $\phi \geq 0$. Then we apply Lemma 3.6 with $w = w_k \in \mathcal{N}_\lambda^+$ (k large enough such that $\frac{(1-q)C_1}{k} < C_2$), we obtain a sequence of functions $f_k : B_{\epsilon_k}(0) \rightarrow \mathbb{R}$ such that $f_k(0) = 1$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda^+$ for all $w \in B_{\epsilon_k}(0)$. It follows from $w_k \in \mathcal{N}_\lambda$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda$ that

$$\|w_k\|^p - \int_{\Omega} a(x) (w_k)_+^{1-q}(x) dx - \lambda \int_{\Omega} b(x) (w_k)_+^{r+1}(x) dx = 0 \quad (4.7)$$

and

$$f_k^p(w) \|w_k + w\phi\|^p - f_k^{1-q}(w) \int_{\Omega} a(x) (w_k + w\phi)_+^{1-q}(x) dx - \lambda f_k^{r+1}(w) \int_{\Omega} b(x) (w_k + w\phi)_+^{r+1}(x) dx = 0. \quad (4.8)$$

Choose $0 < \rho < \epsilon_k$, and $w = \rho v$ with $\|v\| < 1$ then we find $f_k(w)$ such that $f_k(0) = 1$ and $f_k(w)(w_k + w\phi) \in \mathcal{N}_\lambda^+$ for all $w \in B_\rho(0)$. Also we will use the following notation:

$$w_k(x, y) := |w_k(x) - w_k(y)|^{p-2} (w_k(x) - w_k(y)).$$

Lemma 4.3 For $\lambda \in (0, \Lambda)$ we have $|\langle f'_k(0), v \rangle|$ is finite for every $0 \leq v \in C_{X_0}$ with $\|v\| \leq 1$.

Proof. From (4.7) and (4.8) we have that

$$\begin{aligned}
0 &= [f_k^p(w) - 1] \|w_k + w\phi\|^p + \|w_k + w\phi\|^p - \|w_k\|^p \\
&\quad - [f_k^{1-q}(w) - 1] \int_{\Omega} a(x)(w_k + w\phi)_+^{1-q} dx - \int_{\Omega} a(x)[((w_k + w\phi)_+^{1-q} - (w_k)_+^{1-q})] dx \\
&\quad - \lambda [f_k^{r+1}(w) - 1] \int_{\Omega} b(x)(w_k + w\phi)_+^{r+1} dx - \lambda \int_{\Omega} b(x)[((w_k + w\phi)_+^{r+1} - (w_k)_+^{r+1})] dx \\
&\leq [f_k^p(\rho v) - 1] \|w_k + \rho v\phi\|^p + \|w_k + \rho v\phi\|^p - \|w_k\|^p - [f_k^{1-q}(\rho v) - 1] \int_{\Omega} a(x)(w_k + \rho v\phi)_+^{1-q} dx \\
&\quad - \lambda [f_k^{r+1}(\rho v) - 1] \int_{\Omega} b(x)(w_k + \rho v\phi)_+^{r+1} dx - \lambda \int_{\Omega} b(x)[((w_k + \rho v\phi)_+^{r+1} - (w_k)_+^{r+1})] dx,
\end{aligned}$$

since

$$(w_k + \rho v\phi)_+^{1-q}(x) - (w_k)_+^{1-q}(x) = \begin{cases} (w_k + \rho v\phi)^{1-q}(x) - (w_k)^{1-q}(x) & \text{if } w_k \geq 0 \\ 0 & \text{if } w_k \leq 0, w_k + \rho v\phi \leq 0 \\ (w_k + \rho v\phi)^{1-q}(x) & \text{if } w_k \leq 0, w_k + \rho v\phi \geq 0, \end{cases} \quad (4.9)$$

we have $\int_{\Omega} a(x)[((w_k + w\phi)_+^{1-q} - (w_k)_+^{1-q})(x)] dx \geq 0$.

Now dividing by $\rho > 0$ and passing to the limit $\rho \rightarrow 0$, we derive that

$$\begin{aligned}
0 &\leq p \langle f'_k(0), v \rangle \|w_k\|^p + p \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n+ps}} dx dy - (1 - q) \langle f'_k(0), v \rangle \int_{\Omega} a(x)(w_k)_+^{1-q} dx \\
&\quad - \lambda(r + 1) \left(\langle f'_k(0), v \rangle \int_{\Omega} b(x)(w_k)_+^{r+1} dx + \int_{\Omega} b(x)(w_k)_+^r v\phi dx \right) \\
&= \langle f'_k(0), v \rangle \left[p \|w_k\|^p - (1 - q) \int_{\Omega} a(x)(w_k)_+^{1-q}(x) dx - \lambda(r + 1) \int_{\Omega} b(x)(w_k)_+^{r+1}(x) dx \right] \\
&\quad + p \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n+ps}} dx dy - \lambda(r + 1) \int_{\Omega} b(x)(w_k)_+^r v\phi dx \\
&= \langle f'_k(0), v \rangle \left[(p - 1 + q) \|w_k\|^p - \lambda(r + q) \int_{\Omega} b(x)(w_k)_+^{r+1}(x) dx \right] - \lambda(r + 1) \int_{\Omega} b(x)(w_k)_+^r v\phi dx \\
&\quad + p \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x - y|^{n+ps}} dx dy. \quad (4.10)
\end{aligned}$$

From (4.4) and (4.10) we know immediately that $\langle f'_k(0), v \rangle \neq -\infty$. Now we show that $\langle f'_k(0), v \rangle \neq +\infty$. Arguing by contradiction, we assume that $\langle f'_k(0), v \rangle = +\infty$. Since

$$\begin{aligned}
|f_k(\rho v) - 1| \|w_k\| + f_k(\rho v) \|\rho v\phi\| &\geq \|[f_k(\rho v) - 1]w_k + \rho v f_k(\rho v)\phi\| \\
&= \|f_k(\rho v)(w_k + \rho v\phi) - w_k\| \quad (4.11)
\end{aligned}$$

and

$$f_k(\rho v) > f_k(0) = 1$$

for sufficiently large k . From the definition of derivative $\langle f'_k(0), v \rangle$, applying equation (4.2) with $w = f_k(\rho v)(w_k + \rho v\phi) \in \mathcal{N}_\lambda^+$, we clearly have that

$$\begin{aligned}
& [f_k(\rho v) - 1] \frac{\|w_k\|}{k} + f_k(\rho v) \frac{\|\rho v\phi\|}{k} \\
& \geq \frac{1}{k} \|f_k(\rho v)(w_k + \rho v\phi) - w_k\| \\
& \geq J_\lambda(w_k) - J_\lambda(f_k(\rho v)(w_k + \rho v\phi)) \\
& = \left(\frac{1}{p} - \frac{1}{1-q} \right) \|w_k\|^p + \lambda \left(\frac{1}{1-q} - \frac{1}{r+1} \right) \int_\Omega b(x)(w_k)_+^{r+1} dx \\
& \quad + \left(\frac{1}{1-q} - \frac{1}{p} \right) f_k^p(\rho v) \|w_k + \rho v\phi\|^p - \lambda \left(\frac{1}{1-q} - \frac{1}{r+1} \right) f_k^{r+1}(\rho v) \int_\Omega b(x)(w_k + \rho v\phi)_+^{r+1} dx \\
& = \left(\frac{1}{1-q} - \frac{1}{p} \right) (\|w_k + \rho v\phi\|^p - \|w_k\|^p) + \left(\frac{1}{1-q} - \frac{1}{p} \right) [f_k^p(\rho v) - 1] \|w_k + \rho v\phi\|^p \\
& \quad - \lambda \left(\frac{1}{1-q} - \frac{1}{r+1} \right) f_k^{r+1}(\rho v) \int_\Omega b(x)[(w_k + \rho v\phi)_+^{r+1} - (w_k)_+^{r+1}](x) dx \\
& \quad - \lambda \left(\frac{1}{1-q} - \frac{1}{r+1} \right) [f_k^{r+1}(\rho v) - 1] \int_\Omega b(x)(w_k)_+^{r+1} dx.
\end{aligned}$$

Dividing by $\rho > 0$ and passing to the limit as $\rho \rightarrow 0$, we can obtain that

$$\begin{aligned}
& \langle f'_k(0), v \rangle \frac{\|w_k\|}{k} + \frac{\|v\phi\|}{k} \geq \left(\frac{p-1+q}{1-q} \right) \langle f'_k(0), v \rangle \|w_k\|^p - \lambda \left(\frac{r+q}{1-q} \right) \int_\Omega b(x)(w_k)_+^r v\phi dx \\
& \quad + \left(\frac{p-1+q}{1-q} \right) \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x-y|^{n+ps}} dx dy - \lambda \left(\frac{r+q}{1-q} \right) \langle f'_k(0), v \rangle \int_\Omega b(x)(w_k)_+^{r+1} dx \\
& = \frac{\langle f'_k(0), v \rangle}{1-q} \left[(p-1+q) \|w_k\|^p - \lambda(r+q) \int_\Omega b(x)(w_k)_+^{r+1}(x) dx \right] \\
& \quad + \left(\frac{p-1+q}{1-q} \right) \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x-y|^{n+ps}} dx dy - \lambda \left(\frac{r+q}{1-q} \right) \int_\Omega b(x)(w_k)_+^r v\phi dx
\end{aligned}$$

that is,

$$\begin{aligned}
\frac{\|v\phi\|}{k} & \geq \frac{\langle f'_k(0), v \rangle}{1-q} \left[(p-1+q) \|w_k\|^p - \lambda(r+q) \int_\Omega b(x)(w_k)_+^{r+1}(x) dx - \frac{(1-q)\|w_k\|}{k} \right] \\
& \quad + \left(\frac{p-1+q}{1-q} \right) \int_Q \frac{w_k(x, y)((v\phi)(x) - (v\phi)(y))}{|x-y|^{n+ps}} dx dy - \lambda \left(\frac{r+q}{1-q} \right) \int_\Omega b(x)(w_k)_+^r v\phi dx \quad (4.12)
\end{aligned}$$

which is impossible because $\langle f'_k(0), v \rangle = +\infty$ and

$$(p-1+q) \|w_k\|^p - \lambda(r+q) \int_\Omega b(x)(w_k)_+^{r+1}(x) dx - \frac{(1-q)\|w_k\|}{k} \geq C_2 - \frac{(1-q)C_1}{k} > 0.$$

In conclusion, $|\langle f'_k(0), v \rangle| < +\infty$. Furthermore (4.4) with $\|w_k\| \leq C_1$ and two inequalities (4.10) and (4.12) also imply that

$$|\langle f'_k(0), v \rangle| \leq C_3$$

for k sufficiently large and a suitable constant C_3 . \square

Lemma 4.4 For each $0 \leq \phi \in C_{X_0}$ and for every $0 \leq v \in X_0$ with $\|v\| \leq 1$, we have $a(x)w_+^{-q}v\phi \in L^1(\Omega)$ and

$$\int_Q \frac{w(x,y)((v\phi)(x) - (v\phi)(y))}{|x-y|^{n+ps}} dx dy - \int_\Omega a(x)w_+^{-q}v\phi dx - \lambda \int_\Omega b(x)w_+^r v\phi dx \geq 0, \quad (4.13)$$

where $w(x,y) = |w(x) - w(y)|^{p-2}(w(x) - w(y))$.

Proof. Applying (4.11) and (4.2) again, we have that

$$\begin{aligned} & [f_k(\rho v) - 1] \frac{\|w_k\|}{k} + f_k(\rho v) \frac{\|\rho v\phi\|}{k} \\ & \geq \frac{1}{k} \|f_k(\rho v)(w_k + \rho v\phi) - w_k\| \\ & \geq J_\lambda(w_k) - J_\lambda(f_k(\rho v)(w_k + \rho v\phi)) \\ & = \frac{1}{p} \|w_k\|^p - \frac{1}{1-q} \int_\Omega a(x)(w_k)_+^{1-q} dx - \frac{\lambda}{r+1} \int_\Omega b(x)(w_k)_+^{r+1} dx - \frac{1}{p} \|f_k(\rho v)(w_k + \rho v\phi)\|^p dx \\ & \quad + \frac{1}{1-q} \int_\Omega a(x)(f_k(\rho v)(w_k + \rho v\phi))_+^{1-q}(x) dx + \frac{\lambda}{r+1} \int_\Omega b(x)(f_k(\rho v)(w_k + \rho v\phi))_+^{r+1}(x) \\ & = -\frac{f_k^p(\rho v) - 1}{p} \|w_k\|^p - \frac{f_k^p(\rho v)}{p} (\|w_k + \rho v\phi\|^p - \|w_k\|^p) \\ & \quad + \frac{f_k^{1-q}(\rho v) - 1}{1-q} \int_\Omega a(x)(w_k + \rho v\phi)_+^{1-q}(x) + \frac{1}{1-q} \int_\Omega a(x)[((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q})(x)] \\ & \quad + \lambda \frac{f_k^{r+1}(\rho v) - 1}{r+1} \int_\Omega b(x)(w_k + \rho v\phi)_+^{r+1}(x) + \frac{\lambda}{r+1} \int_\Omega b(x)[((w_k + \rho v\phi)_+^{r+1} - (w_k)_+^{r+1})(x)]. \end{aligned}$$

Dividing by $\rho > 0$ and passing to the limit $\rho \rightarrow 0^+$, we obtain

$$\begin{aligned} & |\langle f'_k(0), v \rangle| \frac{\|w_k\|}{k} + \frac{\|v\phi\|}{k} \\ & \geq -\langle f'_k(0), v \rangle \|w_k\|^p - \int_Q \frac{w_k(x,y)(\phi(x) - \phi(y))}{|x-y|^{n+ps}} dx dy \\ & \quad + \langle f'_k(0), v \rangle \int_\Omega a(x)(w_k)_+^{1-q}(x) dx + \liminf_{\rho \rightarrow 0^+} \frac{1}{1-q} \int_\Omega \frac{a(x)[((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q})(x)]}{\rho} dx \\ & \quad + \lambda \langle f'_k(0), v \rangle \int_\Omega b(x)(w_k)_+^{r+1} dx + \lambda \int_\Omega b(x)(w_k)_+^r v\phi dx. \\ & = -\langle f'_k(0), v \rangle \left[\|w_k\|^p - \int_\Omega a(x)(w_k)_+^{1-q}(x) dx - \lambda \int_\Omega b(x)(w_k)_+^{r+1}(x) dx \right] \\ & \quad - \int_Q \frac{w_k(x,y)(\phi(x) - \phi(y))}{|x-y|^{n+ps}} dx dy + \lambda \int_\Omega b(x)(w_k)_+^r v\phi dx \\ & \quad + \liminf_{\rho \rightarrow 0^+} \frac{1}{1-q} \int_\Omega \frac{a(x)[(w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q}](x)}{\rho} dx \\ & = - \int_Q \frac{w_k(x,y)(\phi(x) - \phi(y))}{|x-y|^{n+ps}} dx dy + \lambda \int_\Omega b(x)(w_k)_+^r v\phi dx \\ & \quad + \liminf_{\rho \rightarrow 0^+} \frac{1}{1-q} \int_\Omega \frac{a(x)[(w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q}](x)}{\rho} dx. \end{aligned}$$

Then by above inequality, one can see that

$$\liminf_{\rho \rightarrow 0^+} \int_{\Omega} \frac{a(x)[((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q})(x)]}{\rho} dx$$

is finite. Now, using (4.9), we have $a(x)[((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q})(x)] \geq 0$ for all $x \in \Omega$, for all $t > 0$, then by the Fatou Lemma, we have that

$$\begin{aligned} \int_{\Omega} a(x)(w_k)_+^{-q} v \phi dx &\leq \liminf_{\rho \rightarrow 0^+} \frac{1}{1-q} \int_{\Omega} \frac{a(x)[((w_k + \rho v\phi)_+^{1-q} - (w_k)_+^{1-q})(x)]}{\rho} dx \\ &\leq \frac{|\langle f'_k(0), v \rangle| \|w_k\| + \|v\phi\|}{k} - \lambda \int_{\Omega} b(x)(w_k)_+^r v \phi dx + \int_Q \frac{w_k(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy \\ &\leq \frac{C_1 C_3 \|v\| + \|v\phi\|}{k} - \lambda \int_{\Omega} b(x)(w_k)_+^r v \phi dx + \int_Q \frac{w_k(x, y)(v\phi(x) - v\phi(y))}{|x - y|^{n+ps}} dx dy \end{aligned}$$

Again using the Fatou Lemma and the above relation we have

$$\begin{aligned} \int_{\Omega} a(x)w_+^{-q} v \phi dx &\leq \int_{\Omega} \left[\liminf_{k \rightarrow \infty} a(x)(w_k)_+^{-q} v \phi \right] dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} a(x)(w_k)_+^{-q} v \phi dx \\ &= \int_Q \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (v\phi(x) - v\phi(y))}{|x - y|^{n+ps}} dx dy - \lambda \int_{\Omega} b(x)w_+^r v \phi dx, \end{aligned}$$

which completes the proof of Lemma. \square

Corollary 4.5 *For every $0 \leq \phi \in X_0$, we have $a(x)w_+^{-q}\phi \in L^1(\Omega)$, $w_+ > 0$ in Ω and*

$$\int_Q \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a w_+^{-q} \phi dx - \lambda \int_{\Omega} b w_+^r \phi dx \geq 0. \quad (4.14)$$

Proof. Choosing $v \in X_0$ such that $v \geq 0$, $v \equiv l$ in the neighborhood of support of ϕ and $\|v\| \leq 1$, for some $l > 0$ is a constant. Then we note that $\int_{\Omega} a(x)w_+^{-q}\phi dx < \infty$, for every $0 \leq \phi \in C_{X_0}$ which guarantees that $w_+ > 0$ a.e in Ω . Putting this choice of v in (4.13), we have for every $0 \leq \phi \in C_{X_0}$

$$\int_Q \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a w_+^{-q} \phi dx - \lambda \int_{\Omega} b w_+^r \phi dx \geq 0.$$

Hence by density argument, (4.14) holds for every $0 \leq \phi \in X_0$, which completes the proof of the Corollary.

Lemma 4.6 *We show that $w > 0$ and $w \in \mathcal{N}_{\lambda}^+$.*

Proof. Using (4.14) with $\phi = w^-$, we obtain that

$$\begin{aligned} 0 &\leq \int_Q \frac{|w(x) - w(y)|^{p-2} (w(x) - w(y)) (w^-(x) - w^-(y))}{|x - y|^{n+ps}} dx dy \\ &\leq -\|w^-\|^2 - 2 \int_Q \frac{|w(x) - w(y)|^{p-2} w^-(x) w^+(y)}{|x - y|^{n+ps}} dx dy \leq -\|w^-\|^2 \leq 0. \end{aligned}$$

i.e, $w^- = 0$ a.e. So, $w = w^+ > 0$ a.e by Corollary 4.5. Hence $w > 0$ in Ω . Now using (4.14) with $\phi = w$, we obtain that

$$\|w\|^p \geq \int_{\Omega} a(x)w_+^{1-q}(x)dx + \lambda \int_{\Omega} b(x)w_+^{r+1}(x)dx.$$

On the other hand, by the weak lower semi-continuity of the norm, we have that

$$\begin{aligned} \|w\|^p &\leq \liminf_{k \rightarrow \infty} \|w_k\|^p \leq \limsup_{k \rightarrow \infty} \|w_k\|^p \\ &= \int_{\Omega} a(x)w_+^{1-q}(x)dx + \lambda \int_{\Omega} b(x)w_+^{r+1}(x)dx. \end{aligned}$$

Thus

$$\|w\|^p = \int_{\Omega} a(x)w_+^{1-q}(x)dx + \lambda \int_{\Omega} b(x)w_+^{r+1}(x)dx. \quad (4.15)$$

Consequently, $w_k \rightarrow w$ in X_0 and $w \in \mathcal{N}_{\lambda}$. Now from (4.3) it follows that

$$\begin{aligned} (p-1+q)\|w\|^p - \lambda(r+q) \int_{\Omega} b(x)w_+^{r+1}(x)dx \\ = (p-1+q) \int_{\Omega} a(x)w_+^{1-q}(x)dx - \lambda(r-p+1) \int_{\Omega} b(x)w_+^{r+1}(x)dx > 0, \end{aligned}$$

that is, $w \in \mathcal{N}_{\lambda}^+$. □

Lemma 4.7 *Show that w is in fact a positive weak solution of problem (P_{λ}) .*

Proof. Suppose $\phi \in X_0$ and $\epsilon > 0$, then we define $\Psi(x) = (w + \epsilon\phi)_+(x)$. Let $\Omega = \Omega_1 \times \Omega_2$ with

$$\Omega_1 := \{x \in \Omega : w(x) + \epsilon\phi(x) > 0\} \text{ and } \Omega_2 := \{x \in \Omega : w(x) + \epsilon\phi(x) \leq 0\}.$$

Then $\Psi|_{\Omega_1}(x) = (w + \epsilon\phi)(x)$, and $\Psi|_{\Omega_2}(x) = 0$. Decompose

$$Q := (\Omega_1 \times \Omega^c) \cup (\Omega_2 \times \Omega^c) \cup (\Omega^c \times \Omega_1) \cup (\Omega^c \times \Omega_2) \cup (\Omega_2 \times \Omega_1) \cup (\Omega_1 \times \Omega_2) \cup (\Omega_1 \times \Omega_1) \cup (\Omega_2 \times \Omega_2).$$

Let $M(x, y) = w(x, y)((w + \epsilon\phi)^-(x) - (w + \epsilon\phi)^-(y))K(x, y)$, where $w(x, y) = |w(x) - w(y)|^{p-2}(w(x) - w(y))$ and $K(x, y) = \frac{1}{|x-y|^{n+ps}}$. Then we have

1. $\int_{\Omega_1 \times \Omega^c} M(x, y)dxdy = \int_{\Omega^c \times \Omega_1} M(x, y)dxdy = 0.$
2. $\int_{\Omega_2 \times \Omega^c} M(x, y)dxdy = - \int_{\Omega_2 \times \Omega^c} |w(x)|^{p-2}w(x)(w + \epsilon\phi)(x)K(x, y)dxdy.$
4. $\int_{\Omega^c \times \Omega_2} M(x, y)dxdy = - \int_{\Omega^c \times \Omega_2} |w(x)|^{p-2}w(x)(w + \epsilon\phi)(x)K(x, y)dxdy.$
5. $\int_{\Omega_2 \times \Omega_1} M(x, y)dxdy = - \int_{\Omega_2 \times \Omega_1} w(x, y)(w + \epsilon\phi)(x)K(x, y)dxdy.$
6. $\int_{\Omega_1 \times \Omega_2} M(x, y)dxdy = - \int_{\Omega_1 \times \Omega_2} w(x, y)(w + \epsilon\phi)(x)K(x, y)dxdy.$
7. $\int_{\Omega_1 \times \Omega_1} M(x, y)dxdy = 0.$
8. $\int_{\Omega_2 \times \Omega_2} M(x, y)dxdy = - \int_{\Omega_2 \times \Omega_2} w(x, y)((w + \epsilon\phi)(x) - (w + \epsilon\phi)(y))K(x, y)dxdy.$

Putting Ψ into (4.13) and using (4.15), we see that

$$\begin{aligned}
0 &\leq \int_Q \frac{w(x, y)(\Psi(x) - \Psi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}\Psi dx - \lambda \int_{\Omega} b(x)w_+^r\Psi dx \\
&= \int_Q \frac{w(x, y)((w + \epsilon\phi)(x) - (w + \epsilon\phi)(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}(w + \epsilon\phi) dx \\
&\quad - \lambda \int_{\Omega} b(x)w_+^r(w + \epsilon\phi) dx - \int_{\Omega} a(x)w_+^{-q}(w + \epsilon\phi)^- dx - \lambda \int_{\Omega} b(x)w_+^r(w + \epsilon\phi)^- dx \\
&\quad + \int_Q \frac{w(x, y)((w + \epsilon\phi)^-(x) - (w + \epsilon\phi)^-(y))}{|x - y|^{n+ps}} dx dy \\
&= \epsilon \left(\int_Q \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_+^r\phi dx \right) - \int_{\Omega} a(x)w_+^{1-q} dx \\
&\quad + \int_Q \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy + \int_Q \frac{w(x, y)((w + \epsilon\phi)^-(x) - (w + \epsilon\phi)^-(y))}{|x - y|^{n+ps}} dx dy \\
&\quad + \int_{\Omega_2} a(x)w_+^{-q}(w + \epsilon\phi) dx - \lambda \int_{\Omega_2} b(x)w_+^r(w + \epsilon\phi) dx - \lambda \int_{\Omega} b(x)w_+^{1+r} dx \\
&= \epsilon \left(\int_Q \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_+^r\phi dx \right) + \int_{\Omega_2} a(x)w_+^{-q}(w + \epsilon\phi) dx \\
&\quad - 2 \int_{\Omega_2 \times \Omega^c} \frac{|w(x)|^{p-2}w(x)(w + \epsilon\phi)(x)}{|x - y|^{n+ps}} dx dy - 2 \int_{\Omega_2 \times \Omega_1} \frac{w(x, y)(w + \epsilon\phi)(x)}{|x - y|^{n+ps}} dx dy \\
&\quad - 2 \int_{\Omega_2 \times \Omega_2} \frac{w(x, y)((w + \epsilon\phi)(x) - (w + \epsilon\phi)(y))}{|x - y|^{n+ps}} dx dy - \lambda \int_{\Omega_2} b(x)w_+^r(w + \epsilon\phi) dx \\
&= \epsilon \left(\int_Q \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_+^r\phi dx \right) + \int_{\Omega_2} a(x)w_+^{-q}(w + \epsilon\phi) dx \\
&\quad - 2 \int_{\Omega_2 \times \Omega^c} \frac{|w(x)|^p}{|x - y|^{n+ps}} dx dy - 2 \int_{\Omega_2 \times \Omega_1} \frac{w(x, y)w(x)}{|x - y|^{n+ps}} dx dy - 2 \int_{\Omega_2 \times \Omega_2} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \\
&\quad - \lambda \int_{\Omega_2} b(x)w_+^r(w + \epsilon\phi) dx - \epsilon \left(2 \int_{\Omega_2 \times \Omega^c} \frac{|w(x)|^{p-2}w(x)\phi(x)}{|x - y|^{n+ps}} dx dy + 2 \int_{\Omega_2 \times \Omega_1} \frac{w(x, y)\phi(x)}{|x - y|^{n+ps}} dx dy \right. \\
&\quad \left. + 2 \int_{\Omega_2 \times \Omega_2} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy \right) \\
&\leq \epsilon \left(\int_Q \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_+^r\phi dx \right) \\
&\quad - 2 \int_{\Omega_2 \times \Omega_1} \frac{w(x, y)w(x)}{|x - y|^{n+ps}} dx dy + \int_{\Omega_2} a(x)w_+^{-q}(w + \epsilon\phi) dx - 2\epsilon \left(\int_{\Omega_2 \times \Omega^c} \frac{|w(x)|^{p-2}w(x)\phi(x)}{|x - y|^{n+ps}} dx dy \right. \\
&\quad \left. + \int_{\Omega_2 \times \Omega_1} \frac{w(x, y)\phi(x)}{|x - y|^{n+ps}} dx dy + \int_{\Omega_2 \times \Omega_2} \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy \right) - \lambda \int_{\Omega_2} b(x)w_+^r(w + \epsilon\phi) dx \\
&\leq \epsilon \left(\int_Q \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x)w_+^{-q}\phi dx - \lambda \int_{\Omega} b(x)w_+^r\phi dx \right) \\
&\quad + 2\epsilon \left(\int_{\Omega_2 \times \Omega_1} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{p-1}{p}} \left(\int_{\Omega_2 \times \Omega_1} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}} \\
&\quad - 2\epsilon \left[\left(\int_{\Omega_2 \times \Omega^c} \frac{|w(x)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{p-1}{p}} \left(\int_{\Omega_2 \times \Omega^c} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{\Omega_2 \times \Omega_1} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{p-1}{p}} \left(\int_{\Omega_2 \times \Omega_1} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}} \\
& + \left(\int_{\Omega_2 \times \Omega_2} \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{p-1}{p}} \left(\int_{\Omega_2 \times \Omega_2} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{n+ps}} dx dy \right)^{\frac{1}{p}} \Big] \\
& + \epsilon \lambda \epsilon^r \|b\|_{L^{\frac{p_s^*}{p_s^* - r - 1}}(\Omega_2)} \left(\int_{\Omega_2} |\phi|^{p_s^*} dx \right)^{\frac{r+1}{p_s^*}} - \epsilon \lambda \int_{\Omega_2} b(x) (w_+^r \phi)(x) dx.
\end{aligned}$$

Since the measure of the domain of integration $\Omega_2 = \{x \in \Omega \mid (w + \epsilon\phi)(x) \leq 0\}$ tend to zero as $\epsilon \rightarrow 0$, it follows that $\int_{\Omega_2 \times \Omega_1} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} dx dy \rightarrow 0$ as $\epsilon \rightarrow 0$, and similarly $\int_{\Omega_2 \times \Omega^c} \frac{|\phi(x)|^p}{|x - y|^{n+ps}} dx dy$, $\int_{\Omega_2 \times \Omega_2} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{n+ps}} dx dy$, $\lambda \int_{\Omega_2} b(x) w_+^r \phi dx$ and $\lambda \epsilon^r \|b\|_{L^{\frac{p_s^*}{p_s^* - r - 1}}(\Omega_2)} \left(\int_{\Omega_2} |\phi|^{p_s^*} dx \right)^{\frac{r+1}{p_s^*}}$ all are tend to 0 as $\epsilon \rightarrow 0$. Dividing by ϵ and letting $\epsilon \rightarrow 0$, we obtain

$$\int_Q \frac{w(x, y)(\phi(x) - \phi(y))}{|x - y|^{n+ps}} dx dy - \int_{\Omega} a(x) w_+^{-q} \phi dx - \lambda \int_{\Omega} b(x) w_+^r \phi dx \geq 0$$

and since this holds equally well for $-\phi$, it follows that w is indeed a positive weak solution of problem (P_{λ}^+) and hence a positive solution of (P_{λ}) . \square

Lemma 4.8 *There exists a minimizing sequence $\{W_k\}$ in \mathcal{N}_{λ}^- such that $W_k \rightarrow W$ strongly in \mathcal{N}_{λ}^- . Moreover W is a positive weak solution of (P_{λ}) .*

Proof. Using the Ekeland variational principle again, we may find a minimizing sequence $\{W_k\} \subset \mathcal{N}_{\lambda}^-$ for the minimizing problem $\inf_{\mathcal{N}_{\lambda}^-} J_{\lambda}$ such that for $W_k \rightharpoonup W$ weakly in X_0 and pointwise a.e. in Ω . We can repeat the argument used in Lemma 4.2 to derive that when $\lambda \in (0, \Lambda)$

$$(p - 1 + q) \int_{\Omega} a(x) W_+^{1-q}(x) dx - \lambda(r - p + 1) \int_{\Omega} b(x) W_+^{r+1}(x) dx < 0 \quad (4.16)$$

which yields

$$(p - 1 + q) \int_{\Omega} a(x) (W_k)_+^{1-q}(x) dx - \lambda(r - p + 1) \int_{\Omega} b(x) (W_k)_+^{r+1}(x) dx \leq -C_4$$

for k sufficiently large and a suitable positive constant C_4 . At this point we may proceed exactly as in Lemmas 4.3, 4.4, 4.6, 4.7 and corollary 4.5, we conclude that $W > 0$ is the required positive weak solution of problem (P_{λ}^+) . In particular $W \in \mathcal{N}_{\lambda}$. Moreover from (4.16) it follows that

$$\begin{aligned}
& (p - 1 + q) \|W\|^p - \lambda(r + q) \int_{\Omega} b(x) W_+^{r+1}(x) dx \\
& = (p - 1 + q) \left[\int_{\Omega} a(x) W_+^{1-q}(x) dx + \lambda \int_{\Omega} b(x) W_+^{r+1}(x) dx \right] - \lambda(r + q) \int_{\Omega} b(x) W_+^{r+1}(x) dx \\
& = (p - 1 + q) \int_{\Omega} a(x) W_+^{1-q}(x) dx - \lambda(r - p + 1) \int_{\Omega} b(x) W_+^{r+1}(x) dx < 0,
\end{aligned}$$

that is $W \in \mathcal{N}_{\lambda}^-$. \square

Proof of the Theorem 2.2: From Lemmas 4.7, 4.8 and 3.4, we can conclude that the problem (P_λ) has at least two positive weak solutions $w \in \mathcal{N}_\lambda^+$, $W \in \mathcal{N}_\lambda^-$ with $\|W\| > \|w\|$ for any $\lambda \in (0, \Lambda)$. \square

Proof of the Theorem 2.3: For any $W \in \mathcal{N}_\lambda^-$, it follows from Lemma 3.4 that

$$\|W\| > A_\lambda = \Lambda^{\frac{-1}{r-p+1}} \left(\frac{p-1+q}{r+q} \right)^{\frac{1}{r-p+1}} \left(\frac{1}{\|b\|} \right)^{\frac{1}{r-p+1}} (\sqrt[p]{S})^{\frac{r+1}{r-p+1}} \left(\frac{\Lambda}{\lambda} \right)^{\frac{1}{r-p+1}}.$$

Thus by the definition of Λ , and using $\frac{p(r+q)}{(p-1+q)(r-p+1)} - \frac{r+1}{r-p+1} = \frac{1-q}{p-1+q}$, we obtain,

$$\|W\| > \left(1 + \frac{p-1+q}{r-p+1} \right)^{\frac{1}{p-1+q}} \|a\|^{\frac{1}{p-1+q}} \left(\frac{1}{\sqrt[p]{S}} \right)^{\frac{1-q}{p-1+q}} \left(\frac{\Lambda}{\lambda} \right)^{\frac{1}{r-p+1}}.$$

Hence, let $W_\epsilon \in \mathcal{N}_\lambda^-$ be the solution of problem (P_λ) with $r = p-1+\epsilon$, where $\lambda \in (0, \Lambda)$, we have

$$\|W\| > C_\epsilon \left(\frac{\Lambda}{\lambda} \right)^{\frac{1}{\epsilon}}$$

where $C_\epsilon = \left(1 + \frac{p-1+q}{\epsilon} \right)^{\frac{1}{p-1+q}} \|a\|^{\frac{1}{p-1+q}} \left(\frac{1}{\sqrt[p]{S}} \right)^{\frac{1-q}{p-1+q}} \rightarrow \infty$ as $\epsilon \rightarrow 0^+$. This completes the proof. \square

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